Adiabatic approximation and parametric stochastic resonance in a bistable system with periodically driven barrier

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In this paper we compare the analytical adiabatic exponential approximation and exact numerical description of fluctuation-induced transitions in a "quartic" potential with periodically driven barrier. We show that the adiabatic approximation gives an adequate description of the processes in a wide region of parameter space and the accuracy of the approximation improves with increasing noise intensity. For parameter values outside this region, a different kind of resonant activation, which we call parametric stochastic resonance, is observed.

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I. INTRODUCTION

One-dimensional Markov processes are widely used as models for fluctuation-induced transitions in polystable nonlinear systems. In contrast to their relative simplicity, analytical descriptions of these processes are available in only a restricted number of cases. For example, exact analytical descriptions of fluctuation-induced transitions in time-dependent potentials are possible for linear systems. However, almost all real applications imply motion under time dependent nonlinear forces (e.g., different applications in electronics [1–3], stochastic resonance [4], and the ratchet effect [5,6]).

For nonlinear multistable systems the adiabatic approximation may be used to obtain the approximate characteristics of fluctuation-induced transitions [7–9]. This approximation is valid for small driving frequencies, i.e., for frequencies smaller than the noise-induced hopping rates [8]. The solution of the corresponding Fokker-Planck equation may then be approximated as a steady-state distribution with a slow time variation. Another application of the adiabatic approximation has been the calculation of the decay time (Kramer's time [10]) and of the corresponding decay probability [7,8] or mean first passage time (MFPT) [9]. In this case the time-dependent potential is directly substituted into the corresponding characteristics obtained for the static potential. This approach was used in [9] for the case of a time-ramped force where it was shown to be effective for a wide range of parameters. For time-constant potentials, it was also demonstrated in Refs. [11–14] that the temporal evolution of different characteristics of a Markov process (probability and averages) can often be described by an exponential approximation with a good accuracy even for a large noise intensity, if the proper time scale is substituted into the factor of the exponent. This gives hope that the adiabatic approximation with some modifications may be used in a significantly wider range of parameters than before.

The aim of the present paper is twofold. First, we study fluctuation-induced transitions in a symmetric bistable system with a periodically driven barrier to test the limits of applicability of the adiabatic exponential approximation based on exact time characteristics. As a result we find that the adiabatic approximation works well in a range of parameters wide enough for practical applications. Second, we describe a very interesting phenomenon—a kind of "resonant activation"— [15,16], which cannot be described in the framework of the adiabatic approximation and which, in contrast with the usual phenomenon, appears in an overdamped system with a periodically driven potential barrier. This effect manifests itself by an increase of the decay rate of the probability, or, equivalently, by the presence of a minimum in the mean transition time at a particular value of the frequency of the driving signal. These results can be useful for analysis of the noise properties of practical devices such as Josephson junctions [3].

It should be mentioned that in the present paper we consider a case somewhat opposite to the one studied in Refs. [17–19], where the nonadiabatic escape problem was solved analytically in terms of the so-called logarithmic susceptibility in the approximation of small noise intensity and small modulation of the barrier height. In contrast, here we consider the situation of adiabatic driving (characteristic, for instance, for Josephson electronic devices), using a modified adiabatic approximation that is valid for arbitrary noise intensity and arbitrary amplitude of the driving signal.

The paper is organized as follows. In Sec. II we introduce the model of a fluctuation-induced transition in bistable time-oscillating potentials. In Sec. III the adiabatic exponential approximation for this potential, based on exact time characteristics, is applied and the results are compared with direct integration of the corresponding Fokker-Planck equation. We show that for parameter values for which the adiabatic approximation becomes inadequate, the phenomenon of parametric stochastic resonance appears. Finally, in Sec. IV the main results of the paper are summarized.

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II. MODEL

Consider a process of Brownian diffusion in a potential profile

$$\Phi(x,t)=bx^4-a(t)x^2. \quad (1)$$

It is known that the probability density $W(x,t)$ of the Brownian particle in the overdamped limit (Markov process) satisfies the Fokker-Planck equation (FPE)

$$\frac{\partial W(x,t)}{\partial t}=-\frac{\partial G(x,t)}{\partial x}$$
$$=\frac{1}{B}\left\{\frac{\partial}{\partial x}\left[\frac{d\varphi(x,t)}{dx}W(x,t)\right]+\frac{\partial^2 W(x,t)}{\partial x^2}\right\}. \quad (2)$$

Here $G(x,t)$ is the probability current, $B=h/kT$, h is the viscosity (in computer simulations we put $h=1$), T is the temperature, k is the Boltzmann constant, and $\varphi(x,t)=\Phi(x,t)/kT$ is the dimensionless potential profile. The initial and the boundary conditions have the following form:

$$W(x,0)=\delta(x-x_0),\; G(\pm\infty,t)=0. \quad (3)$$

In the following we consider the case in which the potential varies periodically in time, i.e., we take $a(t)=1+\cos(\omega t+\psi)$ in Eq. (1), where ψ is the phase. We shall restrict ourselves for simplicity to the case $\psi=0$, i.e., we suppose that the potential barrier had maximal height which is decreased during the first half of the period.

We are interested in the time evolution of the probability of finding the Brownian particle in the left potential well,

$$P(t)=\int_{-\infty}^{0}W(x,t)dx, \quad (4)$$

as well as in the evolution of the mean coordinate of the particle,

$$m(t)=\langle x(t)\rangle=\int_{-\infty}^{+\infty}xW(x,t)dx \quad (5)$$

(we assume that the particle is initially located in the left minimum). Since in the course of time the potential barrier moves up and down (remaining symmetric in time) one has that the probability of finding a particle in the left minimum will tend to one-half. In the following we shall use the exponential adiabatic approximation to compute the probability and the mean coordinate of the particle as functions of time.

III. ADIABATIC APPROXIMATION AND PARAMETRIC STOCHASTIC RESONANCE

The exponential adiabatic approximation [7,8] was introduced within the context of stochastic resonance and it is based on the concept of the time-dependent escape time $\tau(t)$. If we consider the decay of a metastable state [$P(x_0,0)=1$, $P(x_0,\infty)=0$], we have that, within this approximation, the probability of finding a particle at the time t in the potential minimum has the form [7,8]

$$P(x_0,t)=\exp\left\{-\int_0^t\frac{1}{\tau_K(t')}dt'\right\}, \quad (6)$$

where $\tau_K(t')$ denotes the approximate mean time of decay (Kramers time [10]), obtained for Brownian diffusion in time-constant potentials in the approximation of small noise intensity in comparison with the barrier height. Recently it has been demonstrated [12–14] that the kinetics of averages of Brownian diffusion in time-constant potentials may often be described by an exponential approximation with good accuracy even for small barrier heights (large noise intensity). This suggests adapting the adiabatic approximation to the present potential $\Phi(x,t)=bx^4-a(t)x^2$ by modifying Eq. (6) as

$$P(x_0,t)=\frac{\exp\left\{-\int_0^t[1/\tau_p(t')]dt'\right\}+1}{2}, \quad (7)$$

where the Kramers time has now been replaced by $\tau_p(x_0,t')$, the exact mean time of transition of the probability to the steady-state value $P(x_0,\infty)=1/2$. Following [20], we have that the static mean transition time $\tau_p(x_0)$ over the point of symmetry $x=0$ of the potential is equal to the MFPT [21],

$$\tau_p(x_0)=T(x_0,0)=B\int_{x_0}^{0}e^{\varphi(y)}\int_{-\infty}^{y}e^{-\varphi(x)}dxdy. \quad (8)$$

Similarly, we write the exponential adiabatic approximation for evolution of the mean coordinate as

$$m(x_0,t)=\langle x(t)\rangle=x_0\exp\left\{-\int_0^t\frac{1}{\tau_m(x_0,t')}dt'\right\}, \quad (9)$$

where $\tau_m(x_0,t')$ is the exact characteristic time of evolution of the mean coordinate. For time-constant symmetric potentials the quantity $\tau_m(x_0)$ can be obtained using the approach of Malakhov [22] and is expressed in the following form [14]:

$$\tau_m(x_0)=\frac{B}{x_0}\left\{\int_0^{+\infty}xe^{-\varphi(x)}dx\int_0^{x_0}e^{\varphi(u)}du\right.$$
$$\left.+\int_0^{x_0}xe^{-\varphi(x)}\int_{x_0}^{x}e^{\varphi(u)}dudx\right\}. \quad (10)$$

If $x_0=0$, it is not difficult to check that $\tau_m(x_0)=0$. The limits of validity of the exponential adiabatic approximation may then be investigated by comparing Eqs. (7) and (9) directly with numerical simulations of the Fokker-Planck equation in Eq. (2). In the following for simplicity we fix $b=1$, $a(0)=2$, so that the maximal barrier height $\Delta\Phi_{max}=\Delta\Phi(t=0)=1$. The noise intensity kT is considered to be a free parameter that varies in the range 0.1 to 2. In the numerical calculations described below we always used an initial distribution located exactly in the minimum of the left potential well, $x_0=-1$. With this choice we found that the exponential approximation gives minimal error in the evolution of the mean coordinate (the probability, as an integral characteristic, is rather independent of the location of the initial distribution).

A comparison between the adiabatic exponential approximation (7) (dashed lines) and the results of a computer simulation (solid lines), for different values of the noise intensity, is presented in Figs. 1–3 for the decay probability. These

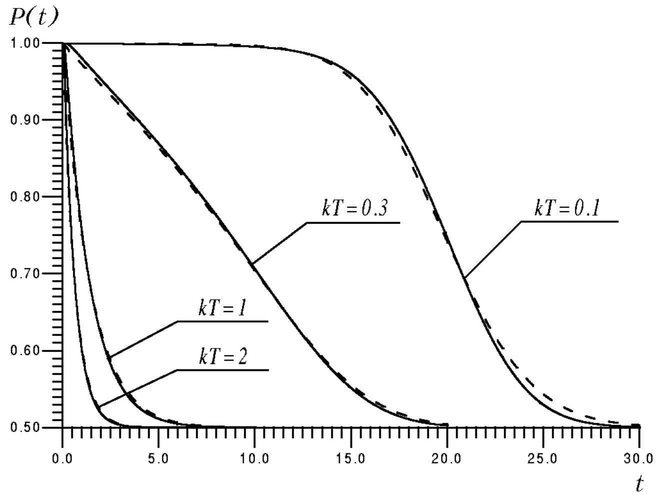

FIG. 1. Evolution of the decay probability for different values of noise intensity, $\omega = 0.1$; solid lines—results of computer simulation, dashed lines—adiabatic approximation (7) (plotted variables are dimensionless).

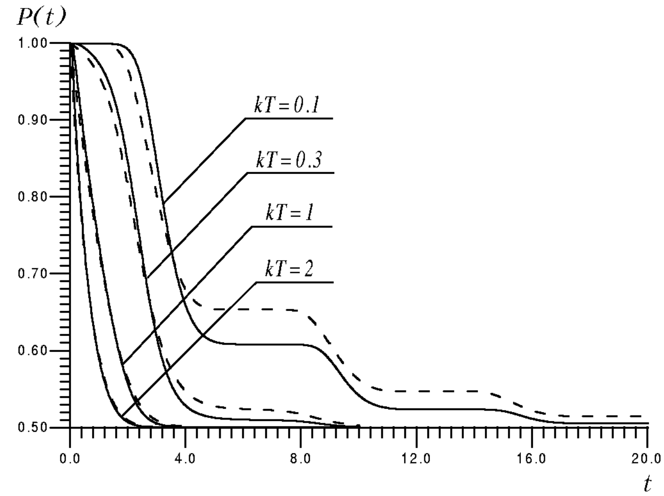

FIG. 3. Evolution of the decay probability for different values of noise intensity, $\omega = 1$; solid lines—results of computer simulation, dashed lines—adiabatic approximation (7) (plotted variables are dimensionless).

figures refer to frequency values, respectively, $\omega = 0.1$, 0.5, and 1.0. We see that for $\omega = 0.1$ (or less), the adiabatic approximation for the decay probability is in perfect agreement with the results of the computer simulation with an error of the order of a few percent (the error is of the same order as the one found for time-constant potentials at the same parameter values [12,13]). On increasing the frequency, the error increases and it depends significantly on the noise intensity. In particular, from Figs. 2 and 3 we see that the deviation of the adiabatic approximation from the numerical results decreases with increasing noise intensity. This fact is more clearly seen at $\omega = 1$, where for small noise intensities the decay of the probability occurs during a few periods of the driver. We remark, however, that even at such high frequencies the adiabatic approximation gives quantitatively good estimates. Similar comparisons for the time evolution of the mean coordinate are reported in Figs. 4–6. Here we see that the deviation of the adiabatic approximation (9) (dashed lines) from the numerical results (continuous curves) is larger than in the previous case. This can be ascribed to a higher sensitivity of the mean coordinate to the location of the initial distribution (namely, at higher frequencies the potential minimum is effectively shifted from the point $x_0 = -1$ so that the initial distribution at the potential slope leads to a higher error). From these results we can say that the modified adiabatic approximation is valid in the low frequency and large noise intensity limit. On the other hand, in the region where the adiabatic approximation becomes inadequate, i.e., high frequencies and small noise intensities, we found an interesting resonant phenomenon that resembles the ''resonant activation'' reported for systems with fluctuating potential barrier in Refs. [15,16]. To describe this phenomenon, consider the curves of the probability evolution at $kT = 0.1$ and for different values of the driving frequency as reported in Fig. 7 (a similar description can be given for the evolution of the mean coordinate). We see that the decay

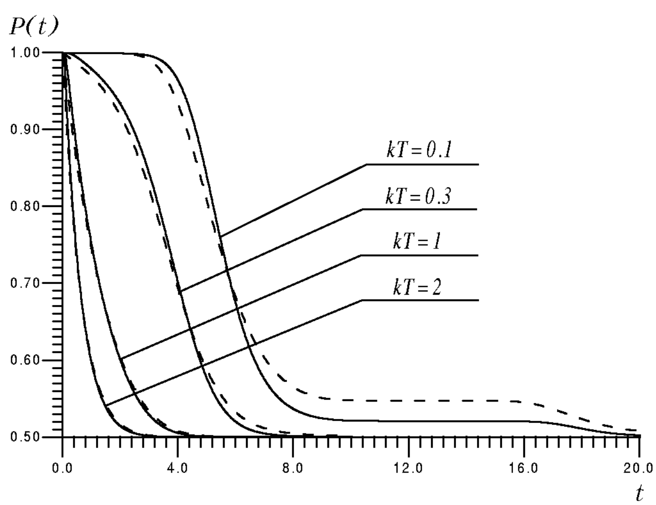

FIG. 2. Evolution of the decay probability for different values of noise intensity, $\omega = 0.5$; solid lines—results of computer simulation, dashed lines—adiabatic approximation (7) (plotted variables are dimensionless).

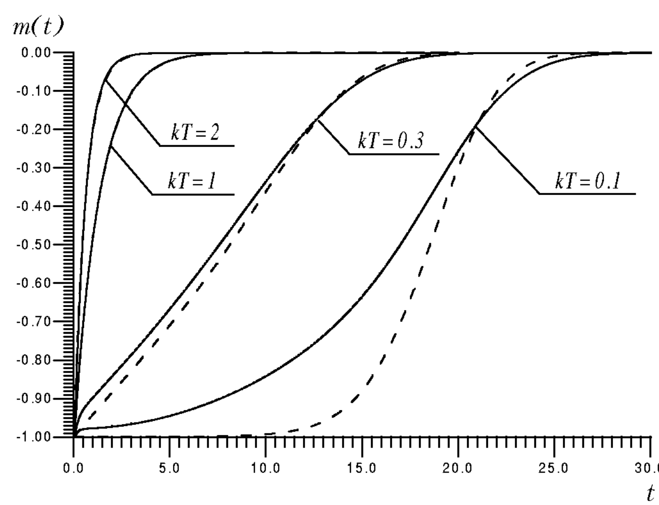

FIG. 4. Evolution of mean coordinate for different values of noise intensity, $\omega = 0.1$; solid lines—results of computer simulation, dashed lines—adiabatic approximation (9) (plotted variables are dimensionless).

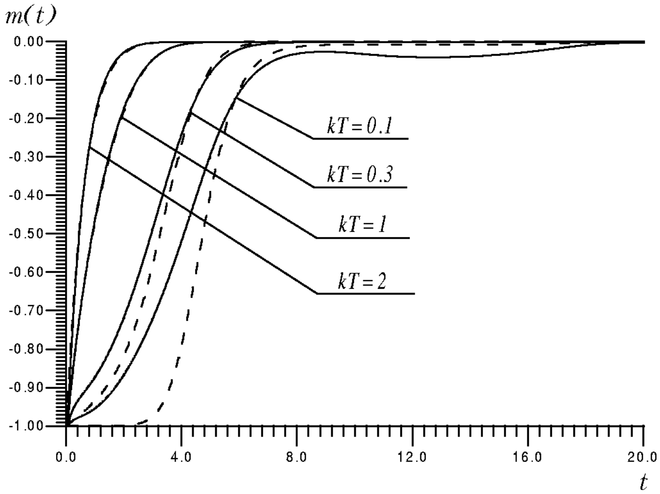

FIG. 5. Evolution of mean coordinate for different values of noise intensity, $\omega=0.5$; solid lines—results of computer simulation, dashed lines—adiabatic approximation (9) (plotted variables are dimensionless).

time of the probability has a minimum at $\omega\approx1$ (an increase or decrease of the frequency away from this value reduces the decay rate of the probability). This resonant phenomenon can be characterized also in terms of the mean transition time $\tau(\omega)$ defined via probability evolution as [22,20,12,13]

$$\tau(\omega)=\frac{\int_0^\infty [P(t)-P(\infty)]dt}{[P(0)-P(\infty)]}. \tag{11}$$

[Note that, in analogy with time-constant potentials [20], $\tau(\omega)$ coincides with the corresponding MFPT for the considered symmetric time-dependent case also, as one can check numerically]. In Fig. 8 we report the mean transition times $\tau(\omega)$ as a function of the frequency for different values of noise intensity. We see that at $kT=0.1$, $\tau(\omega)$ has a minimum at $\omega\approx1$ which almost disappears at large noise intensities (for $kT=1$ the effect has the order of the error). In spite of the similarity with the resonant activation phenomenon, we remark that the potential barrier is not fluctuating according to some probability distribution but oscillates in time, and the system is in the overdamped regime (resonant activation is more characteristic for underdamped systems as discussed in Ref. [8]). The presence of a parametric periodic forcing and of an external noise makes it natural to call this effect ''parametric stochastic resonance.''

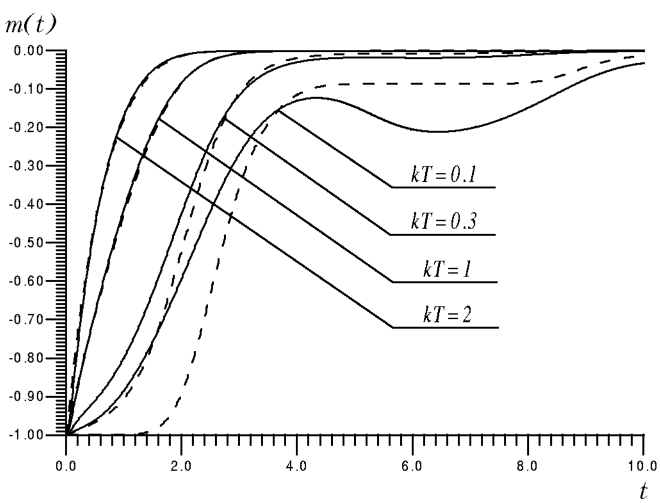

FIG. 6. Evolution of mean coordinate for different values of noise intensity, $\omega=1$; solid lines—results of computer simulation, dashed lines—adiabatic approximation (9) (plotted variables are dimensionless).

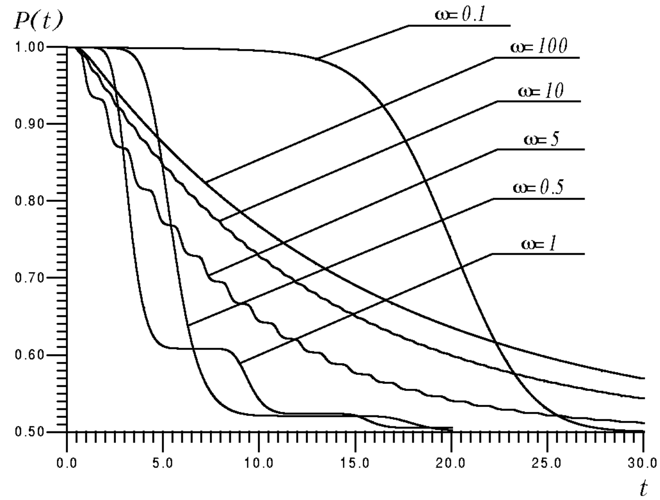

FIG. 7. Evolution of the decay probability for different values of frequency for $kT=0.1$ (plotted variables are dimensionless).

In Fig. 8 the dashed lines denote the results of the adiabatic approximation. We see that although this approximation fails to describe the parametric stochastic resonance, it works well in a wide range of parameters up to frequency $\omega\approx1$ for small noise intensity (note that the error decreases with increase of noise intensity).

We have to mention here that taking the phase of the driving signal into consideration ($\psi\neq0$) will lead to significantly different behavior of the mean transition time (11) in the low frequency regime because the potential barrier height at the initial instant will have a large excursion (from 0 to 1) depending on the phase. However, preliminary results show that if the phase is randomly chosen, after averaging on it,

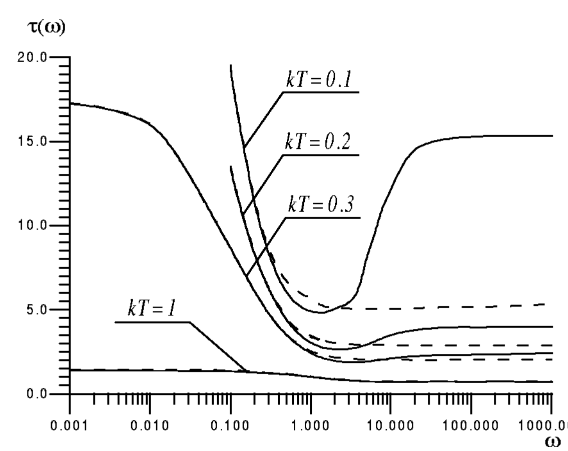

FIG. 8. Mean decay time $\tau(\omega)$ [Eq. (11)] as a function of frequency: solid lines—results of computer simulation; dashed lines—adiabatic approximation (7) (plotted variables are dimensionless).

we get qualitatively similar results (a detailed investigation of this phenomenon will be reported elsewhere).

Finally, we remark that the solid curves $\tau(\omega)$ in Fig. 8 can easily be approximated at the extreme frequency regions. Indeed, for $\omega\to\infty$ (high frequency limit) the cosine term in the coefficient $a(t)=1+\cos(\omega t)$ averages to 1 so that the potential reduces to the time-independent potential $\Phi(x,t)=bx^4-x^2$ and $\tau(\omega)$ is expressed via the MFPT. On the other hand, for $0\leqslant\omega\leqslant1$, we can use the adiabatic approximation (11) and (7) described above.

Unfortunately, an analytical description of these curves for the whole frequency range as a function of the noise intensity is presently lacking, this being an interesting problem for future investigations.

IV. CONCLUSIONS

We have studied fluctuation-induced transitions in a ''quartic'' potential with a periodically driven barrier. We found that the exponential adiabatic approximation based on exact time characteristics gives good estimates in a wide range of parameters and it improves with increase of the noise intensity. This feature makes the adiabatic approximation a useful tool for practical applications such as, for example, Josephson electronic devices [3], in which the driving frequencies are usually rather small compared to the characteristic frequency of the Josephson junction. The surprising thing is that the adiabatic approximation (6) usually used in the limit of small noise intensity, in reality works better (up to higher frequencies of driving signal) for larger noise intensity. This could be due to the fact that the adiabatic approximation [7,8] is based on the concept of instantaneous escape and for higher noise intensity the escape becomes faster.

In the parameter range where the adiabatic approximation fails, we have shown the existence of a different kind of resonant activation which we called parametric stochastic resonance. This phenomenon consists in the appearance of a maximum in the decay rate of the probability or, equivalently, in a minimum in the mean transition time at a particular value of the frequency of the driving signal. The existence of this phenomenon in Josephson devices is presently under investigation [23].

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